

Calabi–Yau threefolds of Borcea–Voisin type and Arithmetic Mirror Symmetry

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Langlands Philosophy and Arithmetic Mirror Symmetry

(Motivic) L-functions of algebraic varieties over \mathbb{Q} (or a number field) are automorphic.

In this talk, I will give examples in support of this philosophy for Calabi–Yau varieties (e.g., certain K3 surfaces, and Calabi–Yau threefolds of Borcea–Voisin type defined over \mathbb{Q}).

I will also discuss arithmetic mirror symmetry for these Calabi–Yau threefolds trying to capture mirror symmetry in terms of zeta-functions and L -series (of some motives).

Calabi–Yau Varieties

Definition: A smooth projective variety X/\mathbb{C} of dimension d is said to be *Calabi–Yau* if

- (1) $H^i(X, \mathcal{O}_X) = 0$ for every i , $0 < i < d$, and
- (2) The canonical bundle \mathcal{K}_X is trivial.

Now introduce Hodge numbers:

$$h^{i,j}(X) := \dim_{\mathbb{C}} H^j(X, \Omega_X^i) \quad \text{for } 0 \leq i, j \leq d$$

Then

$$h^{i,j}(X) = h^{j,i}(X) \quad \text{by complex conjugation}$$

and

$$h^{i,j}(X) = h^{d-i, d-j}(X) \quad \text{by the Serre duality.}$$

Remark In terms of Hodge numbers, X/\mathbb{C} is Calabi–Yau if

- (1) $h^{i,0}(X) = 0$ for every $i, 0 < i < d$, and
- (2) $h^{d,0}(X) = h^{0,d}(X) = \dim_{\mathbb{C}} H^0(X, \Omega_X^d) = \dim_{\mathbb{C}} H^0(X, \mathcal{K}_X) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X) = 1$.

The number $h^{0,d}(X)$ is the *geometric genus* $p_g(X)$ of X .

Numerical characters

- The Betti numbers $B_k(X) := \dim_{\mathbb{C}} H^k(X, \mathbb{C})$.

$$B_k(X) = \sum_{i+j=k} h^{i,j}(X).$$

- The Euler characteristic $E(X) := \sum_{k=0}^{2d} (-1)^k B_k(X)$.

Hodge diamonds

The Hodge numbers of Calabi–Yau varieties are concocted to form the Hodge diamond.

$d = 1$: Elliptic curves

$$h^{1,0} = h^{0,1} = 1$$

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 & B_0 = 1 \\ & 1 & \end{array} \quad \begin{array}{ccc} & & \\ & 1 & \\ & & B_1 = 2 \end{array} \quad \begin{array}{ccc} & & \\ & 1 & \\ & & B_2 = 1 \end{array} \quad \begin{array}{c} E = 0 \end{array}$$

Dimension one Calabi–Yau varieties are elliptic curves.

Examples:

- (1) Smooth cubics in \mathbb{P}^2 , e.g.,

$$X_0^3 + X_1^3 + X_2^3 = 0 \subset \mathbb{P}^2,$$

or its one-parameter deformation:

$$X_0^3 + X_1^3 + X_2^3 - 3\lambda X_0 X_1 X_2 = 0 \subset \mathbb{P}^2.$$

- (2) $y^2 = x^3 + ax + b$ where $a, b \in \mathbb{Q}$ with $4a^3 + 27b^2 \neq 0$.
- (3) Smooth curves in weighted projective 2-spaces $\mathbb{P}^2(q_0, q_1, q_2)$, e.g.,

$$E_2 : X_0^2 + X_1^4 + X_2^4 = 0 \subset \mathbb{P}^2(2, 1, 1)$$

or its one-parameter deformation:

$$X_0^2 + X_1^4 + X_2^4 - 4\lambda X_0 X_1 X_2 = 0 \subset \mathbb{P}^2(2, 1, 1).$$

$$E_3 : X_0^2 + X_1^3 + X_2^6 = 0 \subset \mathbb{P}^2(3,2,1)$$

or its one-parameter deformation:

$$X_0^2 + X_1^3 + X_2^6 - 6\lambda X_0 X_1 X_2 = 0 \subset \mathbb{P}^2(3,2,1).$$

$d = 2$: K3 surfaces

$$h^{1,0} = h^{0,1} = 0, \quad h^{2,0} = h^{0,2} = 1$$

$$\begin{array}{ccc} & 1 & \\ & 0 & 0 \\ 1 & 20 & 1 \\ & 0 & 0 \\ & & 1 \end{array} \quad \begin{array}{l} B_0 = 1 \\ B_1 = 0 \\ B_2 = 22 \\ B_3 = 0 \\ B_4 = 1 \\ E = 24 \end{array}$$

Note that $h^{1,1} = 20$ follows from the formula that
 $2 = \chi(\mathcal{O}_X) = \frac{c_1^2 + c_2}{12}$ where the Chern numbers $c_1 = 0$ and $c_2 = E$.

Examples: (1) Any quartic surface in \mathbb{P}^3 . A typical example is the Fermat quartic:

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0 \subset \mathbb{P}^3,$$

or its one-parameter deformation:

$$X_0^4 + X_1^4 + X_2^4 + X_3^4 - 4\lambda X_0 X_1 X_2 X_3 = 0 \subset \mathbb{P}^3.$$

(2) The 95 families of hypersurface simple $K3$ singularities in weighted projective 3-spaces $\mathbb{P}^3(q_0, q_1, q_2, q_3)$.

(3) A double sextic, e.g., $w^2 = f_6(x, y, z)$ where f_6 is a homogeneous degree 6 polynomial.

(4) Elliptic $K3$ surfaces, e.g., $Y^2 Z = X^3 + A(t) X Z^2 + B(t) Z^3$ with $4A^3(t) + 27B^2(t) \neq 0$.

(5) $\widetilde{A/\pm 1} = \text{Km}(A)$: a Kummer surface, where A is an abelian surface.

$d = 3$: Calabi–Yau threefolds

$$h^{1,0} = h^{0,1} = 0, \quad h^{2,0} = h^{0,2} = 0, \quad h^{3,0} = h^{0,3} = 1, \quad h^{1,1} > 0$$

		1					
		0	0				
		0	$h^{1,1}$	0			
		1	$h^{2,1}$	$h^{1,2}$	1	$B_3 = 2(1 + h^{2,1})$	
		0	$h^{2,2}$	0		$B_4 = h^{2,2}$	
		0	0	0		$B_5 = 0$	
					1	$B_6 = 1$	
						$E = 2(h^{1,1} - h^{2,1})$	

Examples: (1) Quintic threefolds in \mathbb{P}^4 . A typical example is is the Fermat quintic:

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 = 0 \subset \mathbb{P}^4,$$

or its one-parameter deformation:

$$X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5 - 5\lambda X_0 X_1 X_2 X_3 X_4 = 0 \subset \mathbb{P}^4.$$

- (2) Hypersurface or complete intersection Calabi–Yau threefolds in projective/weighted projective 4-spaces $\mathbb{P}^4(q_0, q_1, q_2, q_3, q_4)$.
- (3) A double octic, $t^2 = f_8(x, y, z, w)$ where f_8 is a homogeneous polynomial of degree 8.
- (3) Orbifolds of $E \times S$ where E is an elliptic curve and S a K3 surface, e.g., Calabi–Yau threefolds of Borcea–Voisin type.
- (5) Toric Calabi–Yau threefolds (~ 600 million).

(Topological) Mirror Symmetry Conjecture

Given a Calabi–Yau threefold X , there is a family of Calabi–Yau threefolds \hat{X} such that

$$h^{1,1}(\hat{X}) = h^{2,1}(X), \quad h^{2,1}(\hat{X}) = h^{1,1}(X)$$

so that

$$E(\hat{X}) = -E(X).$$

The largest possible Hodge numbers, or equivalently the Euler characteristic of a Calabi–Yau threefold is not known, but some known examples (~ 600 million) have $h^{1,1}$ or $h^{2,1} \sim 500$.

There are many examples of mirror pairs of Calabi–Yau threefolds in support of the conjecture.

Automorphy (Modularity) Question

Let X be a Calabi–Yau variety of dimension d defined over \mathbb{Q} , say, by vanishing of a finite number of polynomials with coefficients in \mathbb{Q} .

Are there global functions that determine the L -series $L(X, s) =: L_d(X, s)$? More concretely, are there automorphic (modular) forms that determine $L(X, s)$?

We may also ask (motivic) automorphy (modularity) for some motives \mathcal{M} associated to X , i.e., Are $L(\mathcal{M}, s)$ determined by some automorphic functions?

Why should we expect modularity/automorphy?

Modularity Results in the last two decades

- **Dim 1:** Every elliptic curve E over \mathbb{Q} is modular. There is a modular form f of weight 2 on some $\Gamma_0(N)$ such that $L(E, s) = L(f, s)$.
- **Dim 2:** Every singular K3 surface S over \mathbb{Q} is modular. There is a modular form f of weight 3 on some $\Gamma_0(N) + \chi$ or $\Gamma_1(N)$ such that $L(T(S) \otimes \mathbb{Q}_\ell, s) = L(f, s)$.
- **Dim 3:** Every rigid Calabi–Yau threefold X over \mathbb{Q} is modular. There is a modular form f of weight 4 on some $\Gamma_0(N)$ such that $L(X, s) = L(f, s)$.

All these results are obtained by studying 2-dimensional Galois representations associated to Calabi–Yau varieties over \mathbb{Q} in question.

Singular K3 surfaces

Let X be a K3 surface defined over \mathbb{Q} . Let $NS(X)$ denote the Néron–Severi group of X generated by algebraic cycles. It is a free finitely generated abelian group, and

$NS(X) = H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$ so that the rank of $NS(X)$ (called the Picard number of X and denoted by $\rho(X)$) is bounded by 20. Let $T(X) = NS(X)^\perp$ be the orthogonal complement of $NS(X)$ in $H^2(X, \mathbb{Z})$. It has the \mathbb{Z} -rank $22 - \rho(X)$, and is called the group of transcendental cycles on X . We have the decomposition

$$H^2(X, \mathbb{Z}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = (NS(X) \otimes \mathbb{Q}_\ell) \oplus (T(X) \otimes \mathbb{Q}_\ell)$$

and we have the decomposition of the L -series:

$$L_2(X, s) = L(NS(X) \otimes \mathbb{Q}_\ell, s) L(T(X) \otimes \mathbb{Q}_\ell, s).$$

The Tate conjecture is valid for any K3 surface over \mathbb{Q} , which

asserts that

$$H_{\text{et}}^2(\overline{X}, \mathbb{Q}_\ell) \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) = NS(X)_\mathbb{Q}.$$

Thus the L -series $L(NS(X) \otimes \mathbb{Q}_\ell, s)$ is expressed in terms of $\zeta_\mathbb{Q}(s-1)^{\rho(X)}$ if all algebraic cycles are defined over \mathbb{Q} , where $\zeta_\mathbb{Q}(s)$ denotes the Riemann zeta-function. The other extreme is when all algebraic cycles are defined over some algebraic number field \mathbb{L} , then $L(NS(X) \otimes \mathbb{Q}_\ell, s) = \zeta_\mathbb{L}(s-1)^{\rho(X)}$ where $\zeta_\mathbb{L}(s)$ is the Dedekind zeta-function of \mathbb{L} . However, these extreme situations occur very rarely. In general, some algebraic cycles may be defined over \mathbb{Q} , but others are not, in which case, Artin L -function should come into the picture.

Therefore, for K3 surfaces, we will address the modularity/automorphy of the motivic L-series, namely, that of

$$L(T(X) \otimes \mathbb{Q}_\ell, s).$$

Definition: A K3 surface X over \mathbb{Q} is *singular* (or *extremal*) if $\rho(X) = 20$ (so that \mathbb{Z} -rank of $T(X) = 2$).

Theorem (Livné): Every singular K3 surface X over \mathbb{Q} is motivically modular. That is, there is a cusp form f of weight $3 = 2 + 1$ on $\Gamma_0(N)_\chi$ with a character χ or $\Gamma_1(N)$ such that

$$L(T(X) \otimes \mathbb{Q}_\ell, s) = L(f, \chi, s) \quad \text{or } L(f, s).$$

$T(X)$ corresponds to a matrix $\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}$ with $d := 4ac - b^2 > 0$.

Then χ is a Dirichlet character associated to an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$.

There is a compatible system of 2-dimensional ℓ -adic Galois representations associated to $T(X)$, and Livné established the modularity of such representations.

Rigid Calabi–Yau threefolds

Definition: A Calabi–Yau threefold X over \mathbb{Q} is said to be *rigid* if $h^{2,1}(X) = 0$ (so that $B_3(X) = 2$). Thus, the Hodge diamond of any rigid Calabi–Yau threefold is given by

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & & 0 & 0 & & \\ & & 0 & h^{1,1} & 0 & & \\ & 1 & 0 & 0 & 1 & & \\ & & 0 & h^{2,2} & 0 & & \\ & & & 0 & 0 & & \\ & & & & 1 & & \\ & & & & & & \\ \end{array} \quad \begin{array}{l} B_0 = 1 \\ B_1 = 0 \\ B_2 = h^{1,1} \\ B_3 = 2 \\ B_4 = h^{2,2} \\ B_5 = 0 \\ B_6 = 1 \\ E = 2h^{1,1} \end{array}$$

Note that mirror partners of a rigid Calabi–Yau threefold are no longer Calabi–Yau threefolds. Conjecturally, mirrors of rigid Calabi–Yau would be some Fano varieties.

Theorem (Gouv  a–Yui, Dieulefait, 2010, 2011) : Every rigid Calabi–Yau threefold X defined over \mathbb{Q} is modular. That is, there is a cusp form f of weight $4 = 3 + 1$ on $\Gamma_0(N)$ such that

$$L(X, s) = L_3(X, s) = L(f, s)$$

Here N is divisible only by bad primes.

There is a compatible system of 2-dimensional ℓ -adic Galois representations associated to X . Proof relies on the validity of Serre’s conjecture on the residual 2-dimensional Galois representations, proved by Khare–Wintenberger and Kisin (2009).

Modularity of higher dimensional Galois representations

Higher dimensional Galois representations will occur when

- (a) the \mathbb{Z} -rank of $T(X) \geq 3$ (i.e., $\rho(X) \leq 19$) for $d = 2$, or
- (b) $h^{2,1}(X) \geq 1$ (so that $B_3(X) = 2(1 + h^{2,1}(X)) \geq 4$) for $d = 3$.

The (motivic) modularity/automorphy question is currently out of reach in the general setting.

For $d = 2$, we need to have more structures, e.g., lattice polarizations on X , elliptic K3 surfaces, or X has large automorphism groups, etc.

For $d = 3$, we require that X has nice geometric structures, e.g., elliptically or $K3$ fibered, or both, or X has a large automorphism group that $H^3(X, \mathbb{Q}_\ell)$ decomposes into motives of small dimensions.

Calabi–Yau threefolds of Borcea–Voisin type

- Let (E, ι) is an elliptic curve with a non-symplectic involution ι such that the induced map

$$\iota^* : H^{1,0}(E) \rightarrow H^{1,0}(E), \quad \iota^*(\omega_E) = -\omega_E$$

and that $E / \langle \iota \rangle \cong \mathbb{P}^1$. Here ω_E is a unique holomorphic 1-form on E .

- Let (S, σ) is a K3 surface with a non-symplectic involution such that the induced map

$$\sigma^* : H^{2,0}(S) \rightarrow H^{2,0}(S), \quad \sigma^*(\omega_S) = -\omega_S.$$

Decompose $H^2(S, \mathbb{C})$ into the $(+)$ - and $(-)$ -eigenspaces under the action of $\sigma^* : H^2(S, \mathbb{C}) \rightarrow H^2(S, \mathbb{C})$:

$$H^2(S, \mathbb{C}) = H^2(S, \mathbb{C})^+ \oplus H^2(S, \mathbb{C})^-.$$

Set

$$H^2(S, \mathbb{Z})^+ := H^2(S, \mathbb{C})^+ \cap H^2(S, \mathbb{Z})$$

and

$$H^2(S, \mathbb{Z})^- := H^2(S, \mathbb{C})^- \cap H^2(S, \mathbb{Z}).$$

Let

$$r := \text{rank}_{\mathbb{Z}} H^2(S, \mathbb{Z})^+.$$

Then $H^2(S, \mathbb{Z})^+$ and $H^2(S, \mathbb{Z})^-$ have signatures $(1, r - 1)$ and $(2, 20 - r)$ respectively.

Nikulin has classified such pairs (S, σ) of K3 surfaces S with non-symplectic involutions σ , up to deformation.

Theorem (Nikulin, 1979) *There are 75 deformation classes of pairs (S, σ) of K3 surfaces S with non-symplectic involutions σ , and they are completely determined by the triple integers*

$$(r, a, \delta)$$

where r is as above, a is the integer determined by

$$(H^2(S, \mathbb{Z})^+)^{\vee} / H^2(S, \mathbb{Z})^+ \simeq (\mathbb{Z}/2\mathbb{Z})^a.$$

The intersection pairing on $H^2(S, \mathbb{Z})^+$ gives rise to a quadratic form q with values in \mathbb{Q} . We define $\delta = 0$ if q has integer values, and 1 otherwise.

Theorem (Nikulin, 1979) : Let (S, σ) be a pair of $K3$ surface with non-symplectic involution σ . Let S^σ be the fixed locus of S under σ . Then

- (1) If $(r, a, \delta) \neq (10, 10, 0), (10, 8, 0)$, then

$$S^\sigma = C_g \cup L_1 \cup \dots \cup L_k \quad (\text{disjoint union})$$

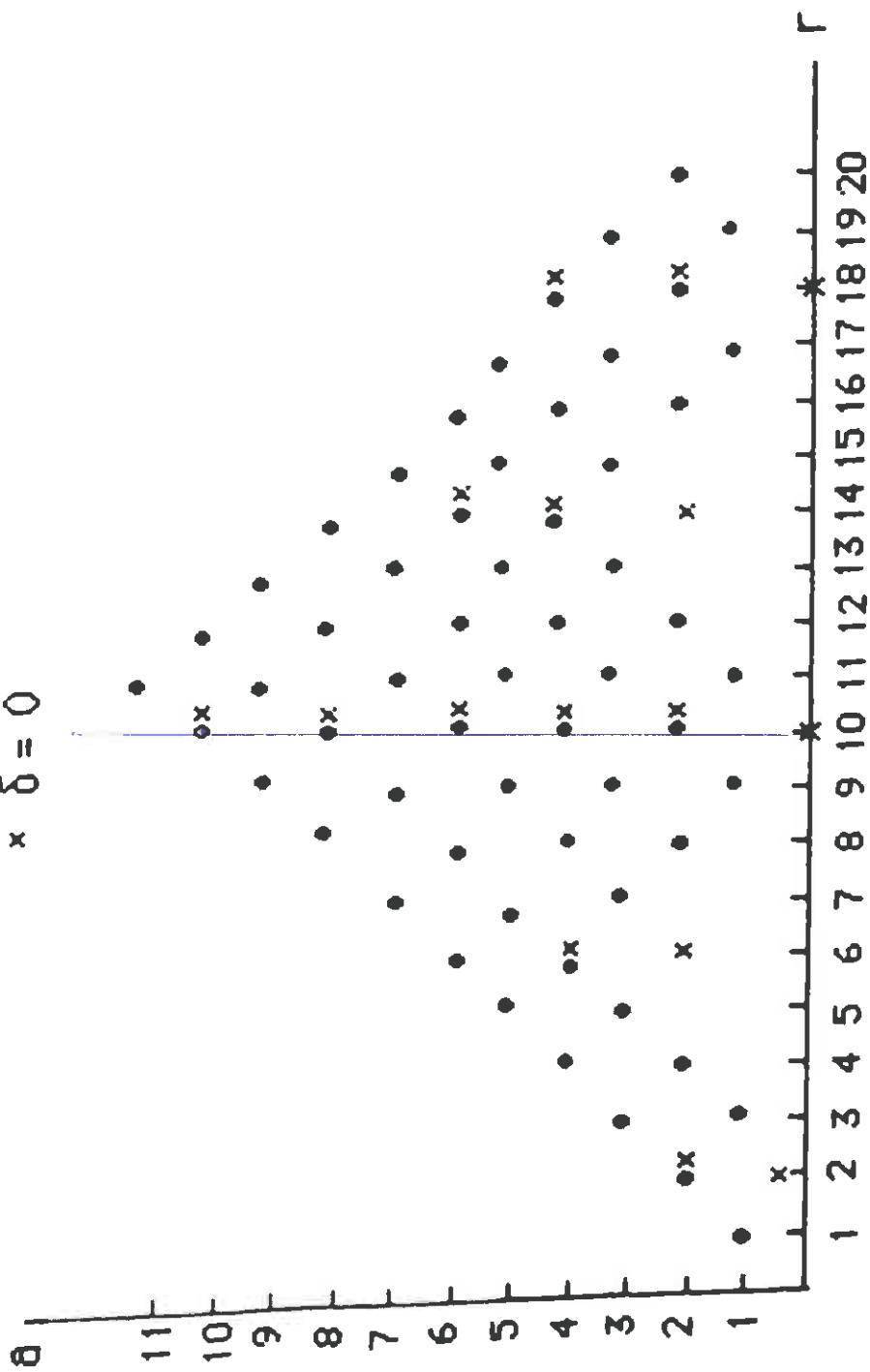
where C_g is a genus $g(\geq 0)$ curve, and $L_i(i = 1, \dots, k)$ are rational curves.

- (2) If $(r, a, \delta) = (10, 10, 0)$, then $S^\sigma = \emptyset$.
- (3) If $(r, a, \delta) = (10, 8, 0)$, then $S^\sigma = C_1 \cup \bar{C}_1$ (disjoint union) where C_1 and \bar{C}_1 are elliptic curves.

Nikulin's pyramid

75 triplets
 (r, a, δ)

- $\delta = 1$
- × $\delta = 0$



Put

$N :=$ the number of components of $S^\sigma = 1 + k$

and

$N' :=$ the sum of genera of components of $S^\sigma = g.$

Note that

$$g = 11 - \frac{1}{2}(r + a), \quad N = 1 + k = 1 + \frac{1}{2}(r - a).$$

Now we will construct Calabi–Yau threefolds of Borcea–Voisin (BV) type. Let (E, ι) and (S, σ) be as above. Take the product $E \times S$. Then the product $\iota \times \sigma$ is an involution on $E \times S$ such that the induced map

$$(\iota \times \sigma)^*: H^{3,0}(E \times S) \rightarrow H^{3,0}(E \times S)$$

is the identity map. Write

$$E^\iota = \{P_1, P_2, P_3, P_4\}$$

and

$$S^\sigma = \{C_1, C_2, \dots, C_N\} \quad \text{with } N = 1 + k.$$

Then the fixed point of $\iota \times \sigma$ consists of

$$P_i \times C_j \quad (i = 1, \dots, 4; j = 1, \dots, N).$$

The involution $\iota \times \sigma$ lifts naturally to an involution on the blow-up of $E \times S$ along $4N$ curves. The quotient $E \times S/\iota \times \sigma$ and its

crepant resolution $\widetilde{E \times S/\iota \times \sigma}$ is our Calabi–Yau threefold of Borcea–Voisin (BV) type, and will be denoted by

$$X = X(r, a, \delta).$$

Note that the exceptional divisors on X are 4 copies of ruled surfaces

$$S^\sigma \times \mathbb{P}^1 := (C_g \times \mathbb{P}^1) \cup (L_1 \times \mathbb{P}^1) \cup \dots \cup (L_k \times \mathbb{P}^1).$$

Theorem (Borcea, Voisin, 1993, 94): *The Hodge numbers of X are given by*

$$h^{1,1}(X) = 11 + 5N - N'$$

$$h^{2,1}(X) = 11 + 5N' - N$$

and

$$E(X) = 12(N - N').$$

(Note that $B_3(X) = 2(12 + 5N' - N)$.)

Mirror symmetry conjecture holds for Calabi–Yau threefolds of Borcea–Voisin type. Mirror symmetry interchanges N and N' , and is inherited from mirror symmetry of K3 surfaces S .

Since any elliptic curve E defined over \mathbb{Q} is modular, the modularity of our Calabi-Yau threefolds $X = X(r, a, \delta)$ depends on the modularity of K3 surface component S . We ought to choose appropriate K3 surfaces for S .

Theorem (Reid 1979, Yonemura 1990): *There are 95 admissible weights (w_0, w_1, w_2, w_3) of hypersurface simple K3 singularities defined by non-degenerate polynomials $F(X_0, X_1, X_2, X_3)$ in weighted projective 3-spaces $\mathbb{P}^3(w_0, w_1, w_2, w_3)$ over \mathbb{Q} .*

Examples: #1: weight $(1, 1, 1, 1)$ $F = X_0^4 + X_1^4 + X_2^4 + X_3^4$.
 #15: weight $(5, 4, 3, 3)$ $F = X_0^3 + X_1^3 X_2 + X_1^3 X_3 + X_2^5 - X_3^5$.
 #95: weight $(7, 5, 3, 2)$

$$F = X_0^2 X_2 + X_0 X_1^2 + X_0 X_3^5 + X_1^3 X_3 + X_1 X_2^4 + X_1 X_3^6 + X_2^5 X_3 + X_2 X_3^7.$$

Among these 95 K3 surfaces, we need to find those with the required involutions.

Theorem (Goto–Livné–Yui, 2014) : *Among the 95 K3 surfaces, 92 have the required non-symplectic involution σ . These 92 pairs (S, σ) realize at least 40 triplets (r, a, δ) of Nikulin. .*

Remark. We have only computed r and a , but not δ . Once δ is computed, the number 40 may increase.

Idea for Proof: Yonemura obtained his defining equations using toric constructions. For each weight, Yonemura wrote down a defining equation using all extremal points of the convex hull determined by the weight. We need to find the required involutions in the 95 families. We were able to find them for the 92 of them, but not yet for the remaining three.

For the second assertion,

- We may remove some monomials keeping in mind Yonemura's condition

(*) *for each i , $0 \leq i \leq 3$, the defining equation must contain a monomial of the form x_i^n or $x_i^n x_j$ ($i \neq j$).*

So if there is a monomial of the form $z_i^n z_j^m$ with $n, m > 1$, we can remove it from the defining equation. We must check that even after removing monomials, a defining equation must remain quasi-smooth. As long as quasi-smoothness is satisfied, we will have

$K3$ surfaces.

- One more condition to keep in mind when removing monomials is that a holomorphic 2-form ω_S should be send to $-\omega_S$ under the involution.
- Also resolution picture should remain the same before and after removing certain monomials.

Theorem (Goto–Livné–Yui): Let (S, σ) be one of the 86 K3 surfaces with involution σ . Then (S, σ) has a defining equation of Delsarte or Fermat type defined over \mathbb{Q} with involution σ . Consequently, it is of CM type (that is, it is realized as a finite Fermat quotient).

Proof: For each weight, with our choice of a defining equation, S is realized as a finite quotient of a Fermat, or a Delsarte surface. The latter surfaces are known to be of CM type. Therefore, (S, σ) is of CM type.

We will illustrate by examples that Calabi–Yau threefolds of Borcea–Voisin type do have birational models defined over \mathbb{Q} .

Example 1: Suppose that S is defined by a hypersurface

$$x_0^2 = f(x_1, x_2, x_3) \subset \mathbb{P}^3(q_0, q_1, q_2, q_3).$$

If q_0 is odd, then $X = \widetilde{E_2 \times S/\iota} \times \sigma$ is birational to a hypersurface defined over \mathbb{Q} :

$$z_0^4 + z_1^4 - f(z_2, z_3, z_4) = 0 \subset \mathbb{P}^4(q_0, q_1, 2q_2, 2q_3).$$

Example 2: Suppose that S is defined by a hypersurface

$$x_0^2 + f(x_1, x_2, x_3) = 0 \subset \mathbb{P}^3(q_0, q_1, q_2, q_3).$$

If q_0 is even but not divisible by 3, then $X = \widetilde{E_3 \times S/\iota} \times \sigma$ is birational to a hypersurface defined over \mathbb{Q} :

$$z_0^3 + z_1^6 - f(z_2, z_3, z_4) = 0 \subset \mathbb{P}^4(2q_0, q_1, 3q_2, 3q_3).$$

Theorem (Goto–Livné–Yui): Let (S, σ) be one of the 86 surfaces represented by a Delsarte surface. Let (E, ι) be an elliptic curve over \mathbb{Q} . Let X be a Calabi–Yau threefold of Borcea–Voisin (BV) type. Then X has a model defined over \mathbb{Q} , and X is (motivically) automorphic.

More precisely,

(a) (S, σ) is (motivically) automorphic, that is,

$$L(S, s) = L(\rho_S, s - 1)L(\chi_S, s)$$

and $L(\chi, s) = L(T(S), s)$ is automorphic. Here ρ_S and χ_S are Galois representations corresponding to $NS(S)$ and $T(S)$, respectively.

(b) X is (motivically) automorphic, that is,

$$L(X, s) = L(\rho_E \otimes \rho_S, s) L(\rho_E \otimes \chi_S, s) L(J(C_g), s - 1)^4$$

where ρ_E is the Galois representation corresponding to E , and $J(C_g)$ is the Jacobian variety of C_g , which is again of CM type.

The L-series $L(\rho_E \otimes \chi_S, s)$ and $L(J(C_g), s - 1)$ are both automorphic.

Arithmetic Mirror Symmetry

The 95 K3 surfaces of Reid and Yonemura are not closed under mirror symmetry of K3 surfaces.

Lemma (Belcastro): *Among the 95 K3 surfaces, the 57 K3 surfaces S have non-symplectic involution σ acting as -1 on $H^{2,0}(S)$, and all 57 have mirror partners S^\vee equipped with non-symplectic involution σ^\vee acting as -1 on $H^{2,0}(S^\vee)$.*

Here if pair (S, σ) of a K3 surface with non-symplectic involution σ corresponds to a Nikulin's triple (r, a, δ) , then a mirror pair (S^\vee, σ^\vee) corresponds to the triplet $(20 - r, a, \delta)$.

Definition:

(1) For a pair (S, σ) of $K3$ surface with a non-symplectic involution σ , we call $T(S)^{\sigma=-1} \otimes \mathbb{Q}_\ell \subset H^2(S, \mathbb{Q}_\ell)$ the $K3$ -motive, and denoted by \mathcal{M}_S . This is the unique motive with $h^{0,2}(\mathcal{M}_S) = 1$.

(2) We will call the submotive $H^1(E, \mathbb{Q}_\ell)^{\iota=-1} \otimes (T(S)^{\sigma=-1} \otimes \mathbb{Q}_\ell)$ of $H^3(X, \mathbb{Q}_\ell)$ the *Calabi–Yau motive* of X , and denoted by \mathcal{M}_X .

Mirror symmetry for Calabi–Yau threefolds is not the correspondence for one Calabi–Yau threefold to another, rather to a family of Calabi–Yau threefolds. We will consider particular member of this mirror family and compare the L -series of Calabi–Yau motives.

Example 1: Let $E = E_2 : y_0^2 = y_1^4 + y_2^4 \subset \mathbb{P}^2(2, 1, 1)$. Let S_0 be a (quasi-smooth) K3 surface given by

$$S_0 : x_0^2 = x_1^3 + x_2^7 + x_3^{42} \subset \mathbb{P}^3(21, 14, 6, 1)$$

of degree 42. S_0 has a non-symplectic involution $\sigma(x_0) = -x_0$. Let S be the minimal resolution of S_0 . Then S corresponds to the triplet $(10, 0, 0)$ of Nikulin. So its mirror S^\vee also corresponds to the triplet $(10, 0, 0)$. The fixed locus $S^\sigma = C_6 \cup L_1 \cup \dots \cup L_5$. Also S is of CM type as it is dominated by the Fermat surface of degree 42. The group of transcendental cycles $T(S)$ corresponds to the cyclotomic field $\mathbb{K} = \mathbb{Q}(\zeta_{42})$ with $[\mathbb{K} : \mathbb{Q}] = \varphi(42) = 12 = 22 - 10$. Then $T(S)$ is automorphic.

Now the Calabi–Yau threefold $X = E_2 \times \widetilde{S/\iota} \times \sigma$ has a birational model defined over \mathbb{Q} :

$$X : z_0^4 + z_1^4 = z_2^3 + z_3^7 + z_4^{42} \subset \mathbb{P}^4(21, 21, 28, 12, 2)$$

of degree 84. Since both E_2 and S are of CM type, X is also of CM type. The Hodge numbers are given by

$$h^{1,1}(X) = 35, h^{21}(X) = 35.$$

So X is own mirror.

Example 2: Let $E = E_2$ and S_0 be a (quasi-smooth) K3 surface given by

$$S_0 : x_0^2 = x_1^3x_2 + x_1^3x_2^2 + x_2^7 - x_3^{14} \subset \mathbb{P}^3(7, 4, 2, 1)$$

of degree 14. S_0 has a non-symplectic involution $\sigma(x_0) = -x_0$. Its minimal resolution S corresponds to the triplet $(7, 3, 0)$ of Nikulin. Remove the monomial $x_1^3x_2^2$ from the defining equation for S_0 , we get

$$S_0 : x_0^2 = x_1^3x_2 + x_2^7 - x_3^{14}$$

which makes S_0 of CM type. It is dominated by the Fermat surface of degree $42 = lcm(3, 2, 14)$. Then $T(S)$ is automorphic.

Now the Calabi–Yau threefold $X = \widetilde{E_2 \times S/\iota} \times \sigma$ has a birational model defined over \mathbb{Q} :

$$X : z_0^4 + z_1^4 = z_2^3 z_3 + z_3^7 - z_4^{14} \subset \mathbb{P}^4(7, 7, 8, 4, 2)$$

of degree 28. The Hodge numbers are given by

$$h^{1,1}(X) = 20, h^{2,1}(X) = 38.$$

The Calabi–Yau motive \mathcal{M}_X has dimension $24 = \varphi(84)$. It is automorphic.

To find a mirror family, we look for a mirror S^\vee of S . We may take for S^\vee the K3 surface defined by

$$S^\vee : x_0^2 = x_1^3 + x_1x_2^7 + x_2^9x_3^2 + x_3^{13} \subset \mathbb{P}^3(21, 14, 4, 3)$$

of degree 42. It has a non-symplectic involution $x_0 \mapsto -x_0$. The pair (S^\vee, σ^\vee) corresponds to the triplet $(13, 3, 0)$. Removing the monomial $x_2^9x_3^2$ we can make S_0 of CM type. Since $\text{lcm}(2, 7, 4) = 28$ $T(S^\vee)$ corresponds to the cyclotomic field $\mathbb{Q}(\zeta_{28})$ of degree $\varphi(28) = 12$. Then $T(S^\vee)$ is automorphic.

A candidate for a mirror family X^\vee has a birational model over \mathbb{Q}

$$X^\vee : z_0^4 + z_1^4 = z_2^3 + z_2 z_3^7 + z_4^{14} \subset \mathbb{P}^4(21, 21, 28, 8, 6)$$

of degree 84. The Hodge numbers are given by

$$h^{1,1}(X^\vee) = 38, h^{2,1}(X^\vee) = 20.$$

We pass from $\mathbb{Q}(\zeta_{28})$ to $\mathbb{Q}(\zeta_{56})$ to take $H^1(E_2)$ into account. Then the Calabi–Yau motive \mathcal{M}_{X^\vee} has dimension $24 = \varphi(56)$. It is automorphic.

In fact, for these examples,

$$L(\mathcal{M}_X, s) = L(\mathcal{M}_{X^\vee}, s)$$

that is, the L-series of the Calabi–Yau motives remain invariant under mirror symmetry.

This phenomenon is valid for many examples of mirror pairs of Calabi–Yau threefolds of Borcea–Voisin type.